

# Tilings with $n$ -Dimensional Chairs and their Applications to Asymmetric Codes

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**Abstract**—An  $n$ -dimensional chair consists of an  $n$ -dimensional box from which a smaller  $n$ -dimensional box is removed. A tiling of an  $n$ -dimensional chair has two nice applications in some memories using asymmetric codes. The first one is in the design of codes which correct asymmetric errors with limited-magnitude. The second one is in the design of  $n$  cells  $q$ -ary write-once memory codes. We show an equivalence between the design of a tiling with an integer lattice and the design of a tiling from a generalization of splitting (or of Sidon sequences). A tiling of an  $n$ -dimensional chair can define a perfect code for correcting asymmetric errors with limited-magnitude. We present constructions for such tilings and prove cases where perfect codes for these type of errors do not exist.

**Index Terms**—Asymmetric limited-magnitude errors, lattice,  $n$ -dimensional chair, perfect codes, splitting, tiling, WOM codes

## I. INTRODUCTION

Storage media which are constrained to change of values in any location of information only in one direction were constructed throughout the last fifty years. From the older punch cards to later optical disks and modern storage such as flash memories, there was a need to design coding which enables the values of information to be increased but not to be decreased. These kind of storage medias are asymmetric memories. We will call the codes used in these medias, asymmetric codes. Some of these memories behave as write-once memories (or WOMs in short) and coding for them was first considered in the seminal work of Rivest and Shamir [19]. This work initiated a sequence of papers on this topic, e.g. [6], [9], [10], [32], [37].

The emerging new storage media of flash memory raised many new interesting problems. Flash memory is a nonvolatile reliable memory with high storage density. Its relatively low cost makes it the ideal memory to replace the magnetic recording technology in storage media. A multilevel flash cell is electronically programmed into  $q$  threshold levels which can be viewed as elements of the set  $\{0, 1, \dots, q-1\}$ . Raising the charge level of a cell is an easy operation, but reducing the charge level of a single cell requires to erase the whole block to which the cell belongs. This makes the reducing of a charge level to be a complicated, slow, and unwanted operation. Hence, the cells of the flash memory act as an asymmetric memory as long as blocks are not erased. This has motivated new research work on WOMs, e.g. [5], [22], [31], [34], [36].

Moreover, usually in programming of the cells, we let the charge level in a single cell of a flash memory only to be

raised, and hence the errors in a single cell will be asymmetric. Asymmetric error-correcting codes were subject to extensive research due to their applications in coding for computer memories [18]. The errors in a cell of a flash memory are a new type of asymmetric errors which have limited-magnitude. Errors in this model are in one direction and are not likely to exceed a certain limit. This means that a cell in level  $i$  can be raised by an error to level  $j$ , such that  $i < j \leq q-1$  and  $j-i \leq \ell \leq q-1$ , where  $\ell$  is the error limited-magnitude. Asymmetric error-correcting codes with limited-magnitude were proposed in [1] and were first considered for nonvolatile memories in [3], [4]. Recently, several other papers have considered the problem, e.g. [7], [8], [13], [35].

In this work we will consider a solution for both the construction problem of asymmetric codes with limited-magnitude and the coding problem in WOMs. Our proposed solution will use an older concept in combinatorics named tiling. Given an  $n$ -dimensional shape  $S \subset \mathbb{Z}^n$ , a tiling of  $\mathbb{Z}^n$  with  $S$  consists of disjoint copies of  $S$  such that each point of  $\mathbb{Z}^n$  is covered by exactly one copy of  $S$ . Tiling is a well established concept in combinatorics and especially in combinatorial geometry. There are many algebraic methods related to tiling [28] and it is an important topic also in coding theory. For example, perfect codes are associated with tilings, where the related sphere is the  $n$ -dimensional shape  $S$ . Tiling is done with a shape  $S$  and we consider only shapes which form an error sphere for asymmetric limited-magnitude codes or their immediate generalization in  $\mathbb{R}^n$ . The definition of a tiling in  $\mathbb{R}^n$  will be given in Section II.

Two of the most considered shapes for tiling are the cross and the semi-cross [26], [28]. These were also considered in connections to flash memories [21]. In this paper we will consider another shape which will be called in the sequel an  $n$ -dimensional chair. An  $n$ -dimensional chair is an  $n$ -dimensional box from which a smaller  $n$ -dimensional box is removed from one of its corners. This is a generalization of the original concept which is an  $n$ -cube from which one vertex was removed [16]. In other places this shape is called a notched cube [15], [20], [27]. Lattice tiling with this shape will be discussed, regardless of the length of each side of the larger box and the length of each side of the smaller box. We will show an equivalent way to present a lattice tiling, this method will be called a generalized splitting and it generalizes the concepts of splitting defined in [23]; and the concept of  $B_h[\ell]$  sequences defined and used for construction of codes correcting asymmetric errors with limited-magnitude in [13]. We will show two applications of tilings with such a shape. One application is for construction of codes which correct up to  $n-1$  asymmetric limited-magnitude errors with any given magnitude for each cell; and a second application is for constructing WOM

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codes with multiple writing.

In the first part of this work we will consider only tilings with  $n$ -dimensional chairs. In the second part of this work we will consider the applications of tilings with  $n$ -dimensional chairs. The rest of this work is organized as follows. In Section II we define the basic concepts for our presentation of tilings with  $n$ -dimensional chairs. We define the concepts of an  $n$ -dimensional chair and a tiling of the space with a given shape. We present the  $n$ -dimensional chair as a shape in  $\mathbb{R}^n$ . When the  $n$ -dimensional chair consists of unit cubes connected only by unit cubes of smaller dimensions, the  $n$ -dimensional chair can be represented as a shape in  $\mathbb{Z}^n$ . For such a shape we will seek for an integer tiling. We will be interested in this paper only in lattice tiling and when the shape is in  $\mathbb{Z}^n$  only in integer lattice tiling. Two representations for tiling with a shape will be given. The first representation is with a generator matrix for the lattice tiling and the second is by the concept which is called a generalized splitting. We will show that these two representations are equivalent. In Section III we will present a construction for tilings with  $n$ -dimensional chairs based on generalized splitting. The construction will be based on properties of some Abelian groups. In Section IV we will present a construction of tiling with  $n$ -dimensional chairs based on lattices. This construction works on any  $n$ -dimensional chair, while the construction of Section III works only on certain discrete ones. We note that after the paper was written it was brought to our attention that lattice tilings for notched cubes were given in [15], [20], [27]. For completeness and since our proof is slightly different we kept this part in the paper. Tiling with a discrete  $n$ -dimensional chair can be viewed as a perfect code for correction of asymmetric errors with limited-magnitude. In Section V we present the definition for such codes, not necessarily perfect. We also present the necessary definition for such perfect codes. We explain what kind of perfect codes are derived from our constructions and also how non-perfect codes can be derived from our constructions. In Section VI we prove that certain perfect codes for correction of asymmetric errors with limited-magnitude do not exist. In Section VII we will discuss the application of our construction for multiple writing in  $n$  cells  $q$ -ary write-once memory. We conclude in Section VIII.

## II. BASIC CONCEPTS

An  $n$ -dimensional chair  $\mathcal{S}_{L,K} \subset \mathbb{R}^n$ ,  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n) \in \mathbb{R}^n$ ,  $0 < k_i < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ , is an  $n$ -dimensional  $\ell_1 \times \ell_2 \times \dots \times \ell_n$  box from which an  $n$ -dimensional  $k_1 \times k_2 \times \dots \times k_n$  box was removed from one of its corners. Formally, it is defined by

$$\mathcal{S}_{L,K} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i < \ell_i,$$

and there exists a  $j$ ,  $1 \leq j \leq n$ , such that  $x_j < \ell_j - k_j\}$ .

For a given  $n$ -dimensional shape  $\mathcal{S}$  let  $|\mathcal{S}|$  denote the *volume* of  $\mathcal{S}$ . The following lemma on the volume of  $\mathcal{S}_{L,K}$  is an immediate consequence of the definition.

*Lemma 1:* If  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n)$  are two vectors in  $\mathbb{R}^n$ , where  $0 < k_i < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ , then

$$|\mathcal{S}_{L,K}| = \prod_{i=1}^n \ell_i - \prod_{i=1}^n k_i.$$

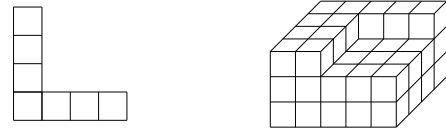


Fig. 1. A semi-cross with  $\ell = 4$  and a 3-dimensional chair with  $L = (5, 4, 3)$  and  $K = (3, 3, 1)$ .

If  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$  then the  $n$ -dimensional chair  $\mathcal{S}_{L,K}$  is a discrete shape and it can be viewed as a collection of connected  $n$ -dimensional unit cubes in which any two adjacent cubes share a complete  $(n-1)$ -dimensional unit cube. In this case the formal definition of the  $n$ -dimensional chair, which considers only points of  $\mathbb{Z}^n$ , is

$$\mathcal{S}_{L,K} = \{(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n : 0 \leq x_i < \ell_i,$$

and there exists a  $j$ ,  $1 \leq j \leq n$ , such that  $x_j < \ell_j - k_j\}$ .

*Remark 1:* It is important to note that if  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n) \in \mathbb{R}^n$  are two integer vectors then the two definitions coincide only if  $\mathcal{S}_{L,K}$  is viewed as a collection of  $n$ -dimensional unit cubes. Special consideration, in the definition, should be given to the boundaries of the cubes, but this is not an issue for the current work.

For  $n = 2$ , if  $\ell_1 = \ell_2 = \ell$  and  $k_1 = k_2 = \ell - 1$ , then the chair coincides with the shape known as a corner (or a semi-cross) [25]. Examples of a two-dimensional semi-cross and a three-dimensional chair are given in Figure 1.

A set  $P \subseteq \mathbb{Z}^n$  is a *packing* of  $\mathbb{Z}^n$  with a shape  $\mathcal{S}$  if copies of  $\mathcal{S}$  placed on the points of  $P$  (in the same relative position of  $\mathcal{S}$ ) are disjoint. A set  $T \subseteq \mathbb{Z}^n$  is a *tiling* of  $\mathbb{Z}^n$  with a shape  $\mathcal{S}$  if it is a packing and the disjoint copies of  $\mathcal{S}$  in the packing cover  $\mathbb{Z}^n$ .

A set  $P \subseteq \mathbb{R}^n$  is a *packing* of  $\mathbb{R}^n$  with a shape  $\mathcal{S}$  if copies of  $\mathcal{S}$  placed on the points of  $P$  (in the same relative position of  $\mathcal{S}$ ) have non-intersecting interiors. The *closure* of a shape  $\mathcal{S} \subset \mathbb{R}^n$  is the union of  $\mathcal{S}$  with its exterior surface. A set  $T \subseteq \mathbb{R}^n$  is a *tiling* of  $\mathbb{R}^n$  with a shape  $\mathcal{S}$  if it is a packing and the closure, of the distinct copies of  $\mathcal{S}$  in the packing, covers  $\mathbb{R}^n$ .

In the rest of this section we will describe two methods to represent a packing (tiling) with a shape  $\mathcal{S}$ . The first representation is with a lattice. In case that  $\mathcal{S}$  is a discrete shape we have a second representation with a splitting sequence.

A *lattice*  $\Lambda$  is an additive subgroup of  $\mathbb{R}^n$ . We will assume that

$$\Lambda \stackrel{\text{def}}{=} \{\lambda_1 V_1 + \lambda_2 V_2 + \dots + \lambda_n V_n : \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{Z}\}$$

where  $\{V_1, V_2, \dots, V_n\}$  is a set of linearly independent vectors in  $\mathbb{R}^n$ . The set of vectors  $\{V_1, V_2, \dots, V_n\}$  is called the *basis* for  $\Lambda$ , and the  $n \times n$  matrix

$$\mathbf{G} \stackrel{\text{def}}{=} \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$$

having these vectors as its rows is said to be the *generator matrix* for  $\Lambda$ . If  $\Lambda \subseteq \mathbb{Z}^n$  then the lattice is called an *integer lattice*.

The *volume* of a lattice  $\Lambda$ , denoted by  $V(\Lambda)$ , is inversely proportional to the number of lattice points per a unit volume. There is a simple expression for the volume of  $\Lambda$ , namely,  $V(\Lambda) = |\det \mathbf{G}|$ .

A lattice  $\Lambda$  is a *lattice packing (tiling)* with a shape  $\mathcal{S}$  if the set of points of  $\Lambda$  forms a packing (tiling) with  $\mathcal{S}$ . The following lemma is well known.

**Lemma 2:** A necessary condition that a lattice  $\Lambda$  defines a lattice packing (tiling) with a shape  $\mathcal{S}$  is that  $V(\Lambda) \geq |\mathcal{S}|$  ( $V(\Lambda) = |\mathcal{S}|$ ). A sufficient condition that a lattice packing  $\Lambda$  defines a lattice tiling with a shape  $\mathcal{S}$  is that  $V(\Lambda) = |\mathcal{S}|$ .

In the sequel, let  $\mathbf{e}_i$  denote the unit vector with an *one* in the  $i$ -th coordinate, let  $\mathbf{0}$  denote the all-zero vector, and let  $\mathbf{1}$  denote the all-one vector. For two vectors  $X = (x_1, x_2, \dots, x_n)$ ,  $Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ , and a scalar  $\alpha \in \mathbb{Z}$ , we define the *vector addition*  $X + Y \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ , and the *scalar multiplication*  $\alpha X \stackrel{\text{def}}{=} (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$ . For a set  $\mathcal{S} \subset \mathbb{R}^n$  and a vector  $U \in \mathbb{R}^n$  the *shift* of  $\mathcal{S}$  by  $U$  is  $U + \mathcal{S} \stackrel{\text{def}}{=} \{U + X : X \in \mathcal{S}\}$ .

Let  $G$  be an Abelian group and let  $\beta = \beta_1, \beta_2, \dots, \beta_n$  be a sequence with  $n$  elements of  $G$ . For every  $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$  we define

$$X \cdot \beta = \sum_{i=1}^n x_i \beta_i,$$

where addition and multiplication are performed in  $G$ .

A set  $\mathcal{S} \subset \mathbb{Z}^n$  *splits* an Abelian group  $G$  with a *splitting sequence*  $\beta = \beta_1, \beta_2, \dots, \beta_n$ ,  $\beta_i \in G$ , for each  $i$ ,  $1 \leq i \leq n$ , if the set  $\{\mathcal{E} \cdot \beta : \mathcal{E} \in \mathcal{S}\}$  contains  $|\mathcal{S}|$  distinct elements from  $G$ . We will call this operation a *generalized splitting*. The splitting defined in [11] and discussed in [12], [23], [24], [26] is a special case of the generalized splitting. It was used for the shapes known as cross and semi-cross [24], [25], and quasi-cross [21]. The  $B_h[\ell]$  sequences defined in [13] and discussed in [13], [14] for construction of codes which correct asymmetric errors with limited-magnitude are also a special case of the generalized splitting. These  $B_h[\ell]$  sequences are modification of the well known Sidon sequences and their generalizations [2]. The generalized splitting also makes generalization for a method discussed by Varshamov [29], [30]. The generalization can be easily obtained, but to our knowledge a general and complete proven theory was not given before.

**Lemma 3:** If  $\Lambda$  is a lattice packing of  $\mathbb{Z}^n$  with a shape  $\mathcal{S} \subset \mathbb{Z}^n$  then there exists an Abelian group  $G$  of order  $V(\Lambda)$ , such that  $\mathcal{S}$  splits  $G$ .

*Proof:* Let  $G = \mathbb{Z}^n / \Lambda$  and let  $\phi : \mathbb{Z}^n \rightarrow G$  be the group homomorphism which maps each element  $X \in \mathbb{Z}^n$  to the coset  $X + \Lambda$ . Clearly,  $|G| = V(\Lambda)$ .

Let  $\beta = \beta_1, \beta_2, \dots, \beta_n$ , be a sequence defined by  $\beta_i = \phi(\mathbf{e}_i)$  for each  $i$ ,  $1 \leq i \leq n$ . Clearly, for each  $X \in \mathbb{Z}^n$  we have  $\phi(X) = X \cdot \beta$ .

Now assume that there exist two distinct elements  $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{S}$ , such that

$$\phi(\mathcal{E}_1) = \mathcal{E}_1 \cdot \beta = \mathcal{E}_2 \cdot \beta = \phi(\mathcal{E}_2) .$$

It implies that

$$\phi(\mathcal{E}_1 - \mathcal{E}_2) = (\mathcal{E}_1 - \mathcal{E}_2) \cdot \beta = \mathcal{E}_1 \cdot \beta - \mathcal{E}_2 \cdot \beta = 0 .$$

Since  $\phi(X) = 0$  if and only if  $X \in \Lambda$  it follows that there exists  $X \in \Lambda$ ,  $X \neq \mathbf{0}$ , such that

$$\mathcal{E}_1 = \mathcal{E}_2 + X .$$

Therefore,  $\mathcal{S} \cap (X + \mathcal{S}) \neq \emptyset$  which contradicts the fact that  $\Lambda$  is a lattice packing of  $\mathbb{Z}^n$  with the shape  $\mathcal{S}$ .

Thus,  $\mathcal{S}$  splits  $G$  with the splitting sequence  $\beta$ . ■

**Lemma 4:** Let  $G$  be an Abelian group and let  $\mathcal{S}$  be a shape in  $\mathbb{Z}^n$ . If  $\mathcal{S}$  splits  $G$  with a splitting sequence  $\beta$  then there exists a lattice packing  $\Lambda$  of  $\mathbb{Z}^n$  with the shape  $\mathcal{S}$ , for which  $V(\Lambda) \leq |G|$ .

*Proof:* Consider the group homomorphism  $\phi : \mathbb{Z}^n \rightarrow G$  defined by

$$\phi(X) = X \cdot \beta .$$

Clearly,  $\Lambda = \ker(\phi)$  is a lattice and the volume of  $\Lambda$ ,  $V(\Lambda) = |\phi(\mathbb{Z}^n)| \leq |G|$ .

To complete the proof we have to show that  $\Lambda$  is a packing of  $\mathbb{Z}^n$  with the shape  $\mathcal{S}$ . Assume to the contrary that there exists  $X \in \Lambda$  such that  $\mathcal{S} \cap (X + \mathcal{S}) \neq \emptyset$ . Hence, there exist two distinct elements  $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{S}$  such that  $\mathcal{E}_1 = \mathcal{E}_2 + X$  and therefore,

$$\phi(\mathcal{E}_1) = \phi(\mathcal{E}_2 + X) = \phi(\mathcal{E}_2) + \phi(X) = \phi(\mathcal{E}_2) .$$

Therefore,  $\mathcal{E}_1 \cdot \beta = \mathcal{E}_2 \cdot \beta$ , which contradicts the fact that  $\mathcal{S}$  splits  $G$  with the splitting sequence  $\beta$ .

Thus,  $\Lambda$  is a lattice packing with the shape  $\mathcal{S}$ . ■

**Corollary 1:** A lattice tiling of  $\mathbb{Z}^n$  with the shape  $\mathcal{S} \subseteq \mathbb{Z}^n$  exists if and only if there exists an Abelian group  $G$  of order  $|\mathcal{S}|$  such that  $\mathcal{S}$  splits  $G$ .

If our shape  $\mathcal{S}$  is not discrete, i.e. cannot be represented by a set of  $n$ -dimensional units cubes, two of which are adjacent only if they share an  $(n - 1)$ -dimensional unit cube, then clearly tiling can be represented with a lattice, but cannot be represented with a splitting sequence. But, if our shape  $\mathcal{S}$  is in  $\mathbb{Z}^n$  then we can use both methods as they were proved to be equivalent. In fact, both methods are complementary. If we consider the matrix  $\mathcal{H} = [\beta_1 \ \beta_2 \ \dots \ \beta_n]$  then the vector  $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$  is contained in the related lattice if and only if  $\mathcal{H}X = 0$ . Therefore,  $\mathcal{H}$  has some similarity to a parity-check matrix in coding theory. The representation of a lattice with its generator matrix seems to be more practical. But, sometimes it is not easy to construct one. Moreover, the splitting sequence has in many cases a nice structure and from its structure the general structure of the lattice can be found. This is the case in the next two sections. In Section III we present two constructions of tilings based on generalized splitting. Even though the second one generalizes the first one, the mathematical structure of the first one has its own beauty and hence both constructions are given. The construction of the lattice, in  $\mathbb{R}^n$ , given in Section IV, was derived based on the structure of the lattices, in  $\mathbb{Z}^n$ , obtained from the construction of the splitting sequences in Section III.

### III. CONSTRUCTIONS BASED ON GENERALIZED SPLITTING

In this section we will present a construction of a tiling with  $n$ -dimensional chairs based on generalized splitting. The  $n$ -dimensional chairs which are considered in this section are discrete, i.e.  $L, K \in \mathbb{Z}^n$ . We start with a construction in which all the  $\ell_i$ 's are equal to  $\ell$ , and all the  $k_i$ 's are equal to  $\ell - 1$ . We generalize this construction to a case in which all the  $k_i$ 's, with a possible exception of one, have multiplicative inverses in the related Abelian group.

For the ring  $G = \mathbb{Z}_q$ , the ring of integers modulo  $q$ , let  $G^*$  be the multiplicative group of  $G$  formed from all the elements of  $G$  which have multiplicative inverses in  $G$ .

*Lemma 5:* Let  $n \geq 2$ ,  $\ell \geq 2$ , be two integers and let  $G$  be the ring of integers modulo  $\ell^n - (\ell - 1)^n$ , i.e.  $\mathbb{Z}_{\ell^n - (\ell - 1)^n}$ . Then,

- (P1)  $\ell - 1$  and  $\ell$  are elements of  $G^*$ .
- (P2)  $\alpha = \ell(\ell - 1)^{-1}$  is an element of order  $n$  in  $G^*$ .
- (P3)  $1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}$  equals to zero in  $G$ .

*Proof:*

- (P1) By definition,  $\ell^n - (\ell - 1)^n$  is zero in  $G = \mathbb{Z}_{\ell^n - (\ell - 1)^n}$ . We also have that  $\ell^n - (\ell - 1)^n = \sum_{i=0}^{n-1} \binom{n}{i} (\ell - 1)^i = 1 + (\ell - 1) \sum_{i=1}^{n-1} \binom{n}{i} (\ell - 1)^{i-1}$ . It follows that  $(\ell - 1)(-\sum_{i=1}^{n-1} \binom{n}{i} (\ell - 1)^{i-1}) = 1$  in  $G$ , and hence,  $\ell - 1 \in G^*$ . Since  $\ell^n - (\ell - 1)^n$  is zero in  $G$ , it follows that  $\ell^n = (\ell - 1)^n$ , and hence  $\ell \in G^*$  if and only if  $\ell - 1 \in G^*$ .
- (P2) Clearly,  $\alpha^n = \ell^n((\ell - 1)^{-1})^n$  and since  $\ell^n = (\ell - 1)^n$ , it follows that  $\alpha^n = (\ell - 1)^n(\ell - 1)^{-n} = 1$ . This also implies that  $\alpha$  has a multiplicative inverse and hence  $\alpha = \ell(\ell - 1)^{-1} \in G^*$ . Now, note that for each  $i$ ,  $1 \leq i \leq n - 1$ , we have  $0 < \ell^i - (\ell - 1)^i < \ell^n - (\ell - 1)^n$ . Therefore,  $\ell^i \neq (\ell - 1)^i$  in  $G$  and hence  $\alpha^i = \ell^i((\ell - 1)^{-1})^i \neq 1$ . Thus, the order of  $\alpha$  in  $G^*$  is  $n$ .
- (P3) Clearly,  $0 = \alpha^n - 1 = (\alpha - 1)(1 + \alpha + \alpha^2 + \dots + \alpha^{n-1})$ . By definition,  $\alpha = \ell(\ell - 1)^{-1}$  and hence  $\alpha(\ell - 1) = \ell$ ,  $\alpha\ell - \alpha = \ell$ ,  $\alpha - \alpha\ell^{-1} = 1$ ,  $\alpha - 1 = \alpha\ell^{-1}$ ,  $\alpha - 1 = (\ell - 1)^{-1}$ . Therefore,  $0 = (\ell - 1)^{-1}(1 + \alpha + \alpha^2 + \dots + \alpha^{n-1})$  which implies that  $1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0$ .

*Theorem 6:* Let  $n \geq 2$ ,  $\ell \geq 2$ , be two integers,  $G = \mathbb{Z}_{\ell^n - (\ell - 1)^n}$ , and  $\alpha = \ell(\ell - 1)^{-1}$ . Then  $\mathcal{S}_{L,K}$ ,  $L = (\ell, \ell, \dots, \ell)$ ,  $K = (\ell - 1, \ell - 1, \dots, \ell - 1)$ , splits  $G$  with the splitting sequence  $\beta = \beta_1, \beta_2, \dots, \beta_n$  defined by

$$\beta_i = \alpha^{i-1}, \quad 1 \leq i \leq n.$$

*Proof:* We will show by induction that every element in  $G$  can be expressed in the form  $\mathcal{E} \cdot \beta$ , for some  $\mathcal{E} \in \mathcal{S}_{L,K}$ .

The basis of induction is  $0 = 0 \cdot \beta$ .

For the induction step we have to show that if  $x \in G$  can be presented as  $x = \mathcal{E} \cdot \beta$  for some  $\mathcal{E} \in \mathcal{S}_{L,K}$  (i.e.  $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathbb{Z}^n$ ,  $0 \leq \varepsilon_i \leq \ell - 1$ ,  $1 \leq i \leq n$ , and for some  $j$ ,  $\varepsilon_j = 0$ ), then also  $x + 1$  can be presented in the same way. In other words,  $x + 1 = \tilde{\mathcal{E}} \cdot \beta$ , where  $\tilde{\mathcal{E}} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_n) \in \mathcal{S}_{L,K}$ .

If  $\varepsilon_1 < \ell - 1$  and there exists  $j \neq 1$  such that  $\varepsilon_j = 0$  then

$$x + 1 = \tilde{\mathcal{E}} \cdot \beta,$$

where  $\tilde{\mathcal{E}} = \mathcal{E} + \mathbf{e}_1$ ,  $0 \leq \tilde{\varepsilon}_i \leq \ell - 1$ ,  $\tilde{\varepsilon}_j = 0$  and the induction step is proved.

If  $\varepsilon_1 = 0$  and there is no  $j \neq 1$  such that  $\varepsilon_j = 0$  then by Lemma 5(P3) we have that  $\sum_{i=1}^n \beta_i = 0$  and hence

$$x + 1 = (\mathcal{E} + \mathbf{e}_1 - \mathbf{1}) \cdot \beta,$$

i.e.  $\tilde{\mathcal{E}} = \mathcal{E} + \mathbf{e}_1 - \mathbf{1}$  is the required element of  $\mathcal{S}_{L,K}$  and the induction step is proved.

Now, assume that  $\varepsilon_1 = \ell - 1$ . Let  $j$ ,  $2 \leq j \leq n$  be the smallest index such that  $\varepsilon_j = 0$ .

$$x + 1 = \ell\beta_1 + \sum_{i=2}^n \varepsilon_i\beta_i.$$

Note that for each  $i$ ,  $1 \leq i \leq n - 1$ ,

$$\ell\beta_i = \ell\ell^{i-1}((\ell - 1)^{-1})^{i-1} = (\ell - 1)\ell^i((\ell - 1)^{-1})^i = (\ell - 1)\beta_{i+1}.$$

Therefore,

$$x + 1 = (\ell - 1 + \varepsilon_2)\beta_2 + \sum_{i=3}^n \varepsilon_i\beta_i.$$

If  $j = 2$  then  $\tilde{\mathcal{E}} = (0, \ell - 1, \varepsilon_3, \dots, \varepsilon_n)$  and the induction step is proved. If  $\varepsilon_2 > 0$ , i.e.  $j > 2$ , then

$$\begin{aligned} x + 1 &= (\varepsilon_2 - 1)\beta_2 + \ell\beta_2 + \sum_{i=3}^n \varepsilon_i\beta_i \\ &= (\varepsilon_2 - 1)\beta_2 + (\ell - 1 + \varepsilon_3)\beta_3 + \sum_{i=4}^n \varepsilon_i\beta_i. \end{aligned}$$

By iteratively continuing in the same manner we obtain

$$x + 1 = \sum_{i=2}^{j-1} (\varepsilon_i - 1)\beta_i + (\ell - 1 + \varepsilon_j)\beta_j + \sum_{i=j+1}^n \varepsilon_i\beta_i$$

and since  $\varepsilon_j = 0$  we have that

$$\tilde{\mathcal{E}} = (0, \varepsilon_2 - 1, \dots, \varepsilon_{j-1} - 1, \ell - 1, \varepsilon_{j+1}, \dots, \varepsilon_n)$$

and the induction step is proved.

Since  $|G| = |\mathcal{S}_{L,K}|$ , it follows that the set  $\{\mathcal{E} \cdot \beta : \mathcal{E} \in \mathcal{S}_{L,K}\}$  has  $|\mathcal{S}_{L,K}|$  elements. ■

*Corollary 2:* For each  $n \geq 2$  and  $\ell \geq 2$  there exists a lattice tiling of  $\mathbb{Z}^n$  with  $\mathcal{S}_{L,K}$ ,  $L = (\ell, \ell, \dots, \ell)$ ,  $K = (\ell - 1, \ell - 1, \dots, \ell - 1)$ .

The next theorem and its proof are generalizations of Theorem 6 and its proof.

*Theorem 7:* Let  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n)$  be two vectors in  $\mathbb{Z}^n$  such that  $0 < k_i < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ . Let  $\tau = \prod_{i=1}^n \ell_i$ ,  $\kappa = \prod_{i=1}^n k_i$ ,  $G = \mathbb{Z}_{\tau - \kappa}$  and assume that for each  $i$ ,  $2 \leq i \leq n$ ,  $k_i \in G^*$ . Then  $\mathcal{S}_{L,K}$  splits  $G$  with the splitting sequence  $\beta = \beta_1, \beta_2, \dots, \beta_n$  defined by

$$\begin{aligned} \beta_1 &= 1 \\ \beta_{i+1} &= k_{i+1}^{-1} \ell_i \beta_i \quad 1 \leq i \leq n - 1. \end{aligned}$$

*Proof:* First we will show that  $k_1\beta_1 = \ell_n\beta_n$ . Since  $\tau - \kappa$  equals zero in  $G$ , it follows that  $\tau = \kappa$  in  $G$  and hence  $k_1 = \ell_1\ell_2 \dots \ell_n k_2^{-1} k_3^{-1} \dots k_n^{-1}$ . Therefore,

$$\begin{aligned} \ell_n\beta_n &= \ell_n k_n^{-1} \ell_{n-1} \beta_{n-1} = \dots \\ &= \ell_n \ell_{n-1} \dots \ell_1 k_n^{-1} k_{n-1}^{-1} \dots k_2^{-1} \beta_1 = k_1 \beta_1. \end{aligned}$$

As an immediate consequence from definition we have that for each  $i$ ,  $1 \leq i \leq n-1$ ,

$$\ell_i \beta_i = k_{i+1} \beta_{i+1}.$$

Next, we will show that

$$(L - K) \cdot \beta = 0. \quad (1)$$

$$\begin{aligned} (L - K) \cdot \beta &= \sum_{i=1}^n (\ell_i - k_i) \beta_i = \sum_{i=1}^n (\ell_i \beta_i - k_i \beta_i) \\ &= \ell_n \beta_n - k_n \beta_n + \sum_{i=1}^{n-1} (k_{i+1} \beta_{i+1} - k_i \beta_i) \\ &= \ell_n \beta_n - k_n \beta_n + k_n \beta_n - k_1 \beta_1 = 0 \end{aligned}$$

Since  $|\mathcal{S}_{L,K}| = |G|$  it follows that to prove Theorem 7, it is sufficient to show that each element in  $G$  can be expressed as  $\mathcal{E} \cdot \beta$ , for some  $\mathcal{E} \in \mathcal{S}_{L,K}$ . The proof will be done by induction.

The basis of induction is  $0 = 0 \cdot \beta$ .

In the induction step we will show that if  $x \in G$  can be presented as  $\mathcal{E} \cdot \beta$  for some  $\mathcal{E} \in \mathcal{S}_{L,K}$  then the same is true for  $x+1$ . In other words,  $x+1 = \tilde{\mathcal{E}} \cdot \beta$ , where  $\tilde{\mathcal{E}} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_n) \in \mathcal{S}_{L,K}$ .

Assume

$$x = \mathcal{E} \cdot \beta,$$

where  $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ ,  $0 \leq \varepsilon_i < \ell_i$  for each  $i$ , and there exists a  $j$  such that  $\varepsilon_j < \ell_j - k_j$ .

If  $\varepsilon_1 < \ell_1 - k_1 - 1$  or if  $\varepsilon_1 < \ell_1 - 1$  and there exists  $j \neq 1$  such that  $\varepsilon_j < \ell_j - k_j$ , then since  $\beta_1 = 1$  it follows that

$$x + 1 = \tilde{\mathcal{E}} \cdot \beta,$$

where  $\tilde{\mathcal{E}} = \mathcal{E} + \mathbf{e}_1$ . Clearly,  $0 \leq \tilde{\varepsilon}_i \leq \ell_i - 1$ ;  $\tilde{\varepsilon}_1 < \ell_1 - k_1$  if  $\varepsilon_1 < \ell_1 - k_1 - 1$  and otherwise  $\tilde{\varepsilon}_j < \ell_j - k_j$ . Hence, the induction step is proved.

If  $\varepsilon_1 = \ell_1 - k_1 - 1$  and there is no  $j \neq 1$  such that  $\varepsilon_j < \ell_j - k_j$  then by (1) we have that  $(L - K) \cdot \beta = 0$  and hence

$$x + 1 = (\mathcal{E} + \mathbf{e}_1 - (L - K)) \cdot \beta,$$

i.e.  $\tilde{\mathcal{E}} = \mathcal{E} + \mathbf{e}_1 - L + K$  is the required element of  $\mathcal{S}_{L,K}$  and the induction step is proved.

Now, assume that  $\varepsilon_1 = \ell_1 - 1$ . Let  $2 \leq j \leq n$  be the smallest index such that  $\varepsilon_j < \ell_j - k_j$ .

$$x + 1 = \ell_1 \beta_1 + \sum_{i=2}^n \varepsilon_i \beta_i = (k_2 + \varepsilon_2) \beta_2 + \sum_{i=3}^n \varepsilon_i \beta_i.$$

If  $j = 2$  then  $\tilde{\mathcal{E}} = (0, k_2 + \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$  and the induction step is proved. If  $\varepsilon_2 \geq \ell_2 - k_2$  then

$$\begin{aligned} x + 1 &= \ell_2 \beta_2 + (\varepsilon_2 - (\ell_2 - k_2)) \beta_2 + \sum_{i=3}^n \varepsilon_i \beta_i \\ &= (\varepsilon_2 - (\ell_2 - k_2)) \beta_2 + (k_3 + \varepsilon_3) \beta_3 + \sum_{i=4}^n \varepsilon_i \beta_i. \end{aligned}$$

By iteratively continuing in the same manner we obtain

$$x + 1 = \sum_{i=2}^{j-1} (\varepsilon_i - (\ell_i - k_i)) \beta_i + (k_j + \varepsilon_j) \beta_j + \sum_{i=j+1}^n \varepsilon_i \beta_i,$$

and since  $\varepsilon_j < \ell_j - k_j$  we have that

$$\tilde{\mathcal{E}} = (0, \varepsilon_2 - \ell_2 + k_2, \dots, \varepsilon_{j-1} - \ell_{j-1} + k_{j-1}, k_j + \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n)$$

is the element of  $\mathcal{S}_{L,K}$ , and the induction step is proved. ■

**Corollary 3:** Let  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n)$  be two vectors in  $\mathbb{Z}^n$  such that  $0 < k_i < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ . Let  $\tau = \prod_{i=1}^n \ell_i$  and assume that  $\gcd(k_i, \tau) = 1$  for at least  $n-1$  of the  $k_i$ 's. Then there exists a lattice tiling of  $\mathbb{Z}^n$  with  $\mathcal{S}_{L,K}$ .

#### IV. TILING BASED ON A LATTICE

Next, we consider lattice tiling of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K} \subset \mathbb{R}^n$ , where  $L = (\ell_1, \ell_2, \dots, \ell_n)$ ,  $K = (k_1, k_2, \dots, k_n) \in \mathbb{R}^n$ . We want to remind again that Mihalis Kolountzakis and James Schmerl pointed on [15], [20], [27], where such tiling can be found. For completeness and since our proof is slightly different we kept this part in the paper. For the proof of the next theorem we need the following lemma.

**Lemma 8:** Let  $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . Then,  $\mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K}) \neq \emptyset$  if and only if  $|x_i| < \ell_i$ , for  $1 \leq i \leq n$ , and there exist integers  $j$  and  $r$ ,  $1 \leq j, r \leq n$ , such that  $x_j < \ell_j - k_j$  and  $-(\ell_r - k_r) < x_r$ .

**Proof:** Assume first that  $\mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K}) \neq \emptyset$ , i.e. there exists  $(a_1, a_2, \dots, a_n) \in \mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K})$ . By the definition of  $\mathcal{S}_{L,K}$  it follows that

$$0 \leq a_i < \ell_i, \quad \text{for each } i, \quad 1 \leq i \leq n, \quad (2)$$

and there exists a  $j$  such that

$$a_j < \ell_j - k_j. \quad (3)$$

Similarly, for  $X + \mathcal{S}_{L,K}$  we have

$$x_i \leq a_i < x_i + \ell_i, \quad \text{for each } i, \quad 1 \leq i \leq n, \quad (4)$$

and there exists an  $r$  such that

$$a_r < x_r + \ell_r - k_r. \quad (5)$$

It follow from (2) and (4) that  $x_i \leq a_i < \ell_i$  and  $-\ell_i \leq a_i - \ell_i < x_i$  for each  $i$ ,  $1 \leq i \leq n$ . Hence,  $|x_i| < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ . It follow from (3) and (4) that  $x_j \leq a_j < \ell_j - k_j$ . It follows from (5) and (2) that  $x_r > a_r - (\ell_r - k_r) \geq -(\ell_r - k_r)$ .

Now, let  $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  such that  $|x_i| < \ell_i$  for each  $i$ ,  $1 \leq i \leq n$ , and there exist  $j, r$  such that  $x_j < \ell_j - k_j$  and  $x_r > -(\ell_r - k_r)$ . Consider the point  $A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ , where  $a_i = \max\{x_i, 0\}$ .

By definition, for each  $i$ ,  $1 \leq i \leq n$ ,

$$0 \leq a_i < \ell_i$$

and  $a_j < \ell_j - k_j$ . Hence,  $A \in \mathcal{S}_{L,K}$ .

Clearly, if  $x_i < 0$  then  $a_i = 0$  and if  $x_i \geq 0$  then  $a_i = x_i$ . In both cases, since  $0 < x_i + \ell_i$ , it follows that we have

$$x_i \leq a_i < x_i + \ell_i.$$

We also have  $0 < x_r + \ell_r - k_r$ , and therefore  $x_r \leq a_r < x_r + \ell_r - k_r$ . Hence,  $A \in X + \mathcal{S}_{L,K}$ .

Thus,  $A \in \mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K})$ , i.e.  $\mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K}) \neq \emptyset$ . ■

The next Theorem is a generalization of Corollary 3.

**Theorem 9:** Let  $L = (\ell_1, \ell_2, \dots, \ell_n) \in \mathbb{R}^n$  and  $K = (k_1, k_2, \dots, k_n) \in \mathbb{R}^n$ ,  $0 < k_i < \ell_i$ , for all  $1 \leq i \leq n$ . Let  $\Lambda$  be the lattice generated by the matrix

$$\mathbf{G}^{\text{def}} \begin{bmatrix} \ell_1 & -k_2 & 0 & 0 & \dots & 0 \\ 0 & \ell_2 & -k_3 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \ell_{n-2} & -k_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \ell_{n-1} & -k_n \\ -k_1 & 0 & \dots & 0 & 0 & \ell_n \end{bmatrix}.$$

Then  $\Lambda$  is a lattice tiling of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K}$ .

*Proof:* It is easy to verify that  $V(\Lambda) = |\det \mathbf{G}| = \prod_{i=1}^n \ell_i - \prod_{i=1}^n k_i = |\mathcal{S}_{L,K}|$ . We will use Lemma 2 to show that  $\Lambda$  is a tiling of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K}$ . For this, it is sufficient to show that  $\Lambda$  is a packing of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K}$ .

Let  $X \in \Lambda$ ,  $X \neq \mathbf{0}$ , and assume to the contrary that  $\mathcal{S}_{L,K} \cap (X + \mathcal{S}_{L,K}) \neq \emptyset$ . Since  $X \in \Lambda$  it follows that there exist integers  $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n = \lambda_0$ , not all zeros, such that  $x_i = \lambda_i \ell_i - \lambda_{i-1} k_i$ , for every  $i$ ,  $1 \leq i \leq n$ . By Lemma 8 we have that for each  $i$ ,  $1 \leq i \leq n$ ,

$$-\ell_i < \lambda_i \ell_i - \lambda_{i-1} k_i < \ell_i,$$

i.e.

$$\frac{\lambda_{i-1} k_i}{\ell_i} - 1 < \lambda_i < \frac{\lambda_{i-1} k_i}{\ell_i} + 1.$$

Since  $\lambda_i$  is an integer it follows that  $\lambda_i = \left\lfloor \frac{\lambda_{i-1} k_i}{\ell_i} \right\rfloor$  or  $\lambda_i = \left\lceil \frac{\lambda_{i-1} k_i}{\ell_i} \right\rceil$ . For each  $i$ ,  $0 \leq i \leq n-1$ , if  $\lambda_i = \rho \geq 0$  then since  $k_{i+1} < \ell_{i+1}$  we have that

$$0 \leq \left\lfloor \frac{\rho k_{i+1}}{\ell_{i+1}} \right\rfloor \leq \lambda_{i+1} \leq \left\lceil \frac{\rho k_{i+1}}{\ell_{i+1}} \right\rceil \leq \rho.$$

Hence,

$$0 \leq \lambda_{i+1} \leq \lambda_i. \quad (6)$$

Similarly, if  $\lambda_i \leq 0$  we have that

$$\lambda_i \leq \lambda_{i+1} \leq 0.$$

If  $\lambda_0 \geq 0$  then by (6) we have

$$\lambda_0 = \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1 \leq \lambda_0,$$

and hence  $\lambda_i = \rho$  for each  $i$ ,  $1 \leq i \leq n$ . Similarly, we have  $\lambda_i = \rho$  for each  $i$ ,  $1 \leq i \leq n$  if  $\lambda_0 \leq 0$ . If  $\rho > 0$  then since  $\rho$  is an integer we have that  $x_i = \rho(\ell_i - k_i) \geq \ell_i - k_i$ , for each  $i$ ,  $1 \leq i \leq n$ . Hence, there is no  $j$  such that  $x_j < \ell_j - k_j$ , which contradicts Lemma 8. Similarly, if  $\rho < 0$  then for each  $i$ ,  $1 \leq i \leq n$ ,  $x_i = \rho(\ell_i - k_i) \leq -(\ell_i - k_i)$ , and hence there is no  $r$  such that  $x_r > -(\ell_r - k_r)$ , which contradicts Lemma 8. Therefore,  $\rho = 0$ , i.e. for each  $i$ ,  $0 \leq i \leq n$ ,  $\lambda_i = 0$ , a contradiction. Hence,  $\Lambda$  is a lattice packing of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K}$ .

Thus, by Lemma 2,  $\Lambda$  is a lattice tiling of  $\mathbb{R}^n$  with  $\mathcal{S}_{L,K}$ . ■

**Remark 2:** Note, that the construction (Theorem 9) is based on lattices covers all the parameters of integers which are not covered in Section III.

## V. ASYMMETRIC ERRORS WITH LIMITED-MAGNITUDE

The first application for a tiling of  $\mathbb{Z}^n$  with an  $n$ -dimensional chair is in construction of codes of length  $n$  which correct asymmetric errors with limited-magnitude.

Let  $Q = \{0, 1, \dots, q-1\}$  be an alphabet with  $q$  letters. For a word  $X = (x_1, x_2, \dots, x_n) \in Q^n$ , the *Hamming weight* of  $X$ ,  $w_H(X)$ , is the number of nonzero entries in  $X$ , i.e.,  $w_H(X) = |\{i : x_i \neq 0\}|$ .

A code  $\mathcal{C}$  of length  $n$  over the alphabet  $Q$  is a subset of  $Q^n$ . A vector  $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is a *t-asymmetric-error with limited-magnitude  $\ell$*  if  $w_H(\mathcal{E}) \leq t$  and  $0 \leq \varepsilon_i \leq \ell$  for each  $1 \leq i \leq n$ . The sphere  $\mathcal{S}(n, t, \ell)$  is the set of all *t-asymmetric-errors with limited-magnitude  $\ell$* . A code  $\mathcal{C} \subseteq Q^n$  can correct *t-asymmetric-errors with limited-magnitude  $\ell$*  if for any two codewords  $X_1, X_2$ , and any two *t-asymmetric-errors with limited-magnitude  $\ell$* ,  $\mathcal{E}_1, \mathcal{E}_2$ , such that  $X_1 + \mathcal{E}_1 \in Q^n$ ,  $X_2 + \mathcal{E}_2 \in Q^n$ , we have that  $X_1 + \mathcal{E}_1 \neq X_2 + \mathcal{E}_2$ .

The size of the sphere  $\mathcal{S}(n, t, \ell)$  is easily computed.

**Lemma 10:**  $|\mathcal{S}(n, t, \ell)| = \sum_{i=0}^t \binom{n}{i} \ell^i$ .

**Corollary 4:**  $|\mathcal{S}(n, n-1, \ell)| = (\ell+1)^n - \ell^n$ .

For simplicity it is more convenient to consider the code  $\mathcal{C}$  as a subset of  $\mathbb{Z}_q^n$ , where all the additions are performed modulo  $q$ . Such a code  $\mathcal{C}$  can be viewed also as a subset of  $\mathbb{Z}^n$  formed by the set  $\{X + qY : X \in \mathcal{C}, Y \in \mathbb{Z}^n\}$ . This code is an *extension*, from  $\mathbb{Z}_q^n$  to  $\mathbb{Z}^n$ , of the code  $\mathcal{C}$ . Note, in this code there is a wrap around (of the alphabet) which does not exist if the alphabet is  $Q$ , as in the previous code.

A linear code  $\mathcal{C}$ , over  $\mathbb{Z}_q^n$ , which corrects *t-asymmetric-errors with limited-magnitude  $\ell$* , viewed as a subset of  $\mathbb{Z}^n$ , is equivalent to an integer lattice packing of  $\mathbb{Z}^n$  with the shape  $\mathcal{S}(n, t, \ell)$ . Therefore, we will call this lattice a *lattice code*.

Let  $\mathcal{A}(n, t, \ell)$  denote the set of lattice codes in  $\mathbb{Z}^n$  which correct *t-asymmetric-errors with limited-magnitude  $\ell$* . A code  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$  is called *perfect* if it forms a lattice tiling with the shape  $\mathcal{S}(n, t, \ell)$ . By Corollary 1 we have

**Corollary 5:** A perfect lattice code  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$  exists if and only if there exists an Abelian group  $G$  of order  $|\mathcal{S}(n, t, \ell)|$  such that  $\mathcal{S}(n, t, \ell)$  splits  $G$ .

A code  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$  is formed as an extension of a code over  $\mathbb{Z}_q^n$ . Assume we want to form a code  $\mathcal{C} \subseteq \Sigma^n$ , where  $\Sigma^{\text{def}} \{0, 1, \dots, \sigma-1\}$ , which corrects *t asymmetric errors with limited-magnitude  $\ell$* . Assume that a construction with a large linear code  $\mathcal{C} \subseteq \Sigma^n$  does not exist. One can take a lattice code  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$  over an alphabet with  $q$  letters  $q > \sigma$ . Then a code over the alphabet  $\Sigma$  is formed by  $\mathcal{C}^{\text{def}} \mathcal{L} \cap \Sigma^n$ . Note that the code  $\mathcal{C}$  is usually not linear. This is a simple construction which always works. Of course, we expect that there will be many alphabets in which better constructions can be found.

There exists a perfect lattice code  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$  for various parameters with  $t = 1$  [13], [14]. Such codes also exist for  $t = n$  and all  $\ell \geq 1$  and for the parameters of the Golay codes and the binary repetition codes of odd length [17].

The existence of perfect codes which correct  $(n-1)$ -asymmetric-errors with limited magnitude  $\ell$  was proved in [14]. The related sphere  $\mathcal{S}(n, n-1, \ell)$  is an  $n$ -dimensional chair  $\mathcal{S}_{L,K}$ , where  $L = (\ell+1, \ell+1, \dots, \ell+1)$  and  $K = (\ell, \ell, \dots, \ell)$ . Sections III and IV provide constructions for such codes with simpler description and simpler proofs that these codes are such perfect codes.

In fact, the constructions in these sections provide tilings of many other related shapes. More than that, there might be scenarios in which different flash cells can have different limited magnitude. For example, if for some cells we want to increase the number of charge levels. In this case we might need a code which correct asymmetric errors with different limited magnitudes for different cells. Assume that for the  $i$ -th cell the limited magnitude is  $\ell_i$ . Our lattice tiling with  $\mathcal{S}_{L,K}$ ,  $L = (\ell_1 + 1, \ell_2 + 1, \dots, \ell_n + 1) \in \mathbb{Z}^n$ ,  $K = (\ell_1, \ell_2, \dots, \ell_n)$ , produces the required perfect code for this scenario.

## VI. NONEXISTENCE OF SOME PERFECT CODES

Next, we ask whether perfect codes, which correct asymmetric errors with limited-magnitude, exist for  $t = n - 2$ . Unfortunately, such codes cannot exist. The proof for this claim is the goal of this section. Most of the proof is devoted to the case in which the limited magnitude  $\ell$  is equal to one. We conclude the section with a proof for  $\ell > 1$ .

For a word  $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ , we define

$$N_+(X) = |\{x_i \mid x_i > 0\}|, \quad N_-(X) = |\{x_i \mid x_i < 0\}|.$$

We say that a codeword  $X \in \mathcal{L}$ ,  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$ , covers a word  $Y \in \mathbb{Z}^n$  if there exists an element  $\mathcal{E} \in \mathcal{S}(n, t, \ell)$  such that  $Y = X + \mathcal{E}$ .

**Lemma 11:** Let  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$ , and assume that there exists  $X = (x_1, x_2, \dots, x_n) \in \mathcal{L}$ ,  $X \neq \mathbf{0}$ , such that  $|x_i| \leq \ell$ , for every  $i$ ,  $1 \leq i \leq n$ . Then,  $N_+(X) \geq t + 1$  or  $N_-(X) \geq t + 1$ .

*Proof:* Let  $X = (x_1, x_2, \dots, x_n) \in \mathcal{L}$ ,  $X \neq \mathbf{0}$ , such that  $|x_i| \leq \ell$ , for every  $i$ ,  $1 \leq i \leq n$ . Assume to the contrary that  $N_+(X) \leq t$  and  $N_-(X) \leq t$ . Let  $\mathcal{E}^+ = (\varepsilon_1^+, \varepsilon_2^+, \dots, \varepsilon_n^+)$  where  $\varepsilon_i^+ = \max\{x_i, 0\}$  and  $\mathcal{E}^- = (\varepsilon_1^-, \varepsilon_2^-, \dots, \varepsilon_n^-)$  where  $\varepsilon_i^- = \max\{-x_i, 0\}$ . Clearly,  $\mathcal{E}^+, \mathcal{E}^- \in \mathcal{S}(n, t, \ell)$  and  $X + \mathcal{E}^- = \mathcal{E}^+$ .

Therefore,  $\mathcal{S}(n, t, \ell) \cap (X + \mathcal{S}(n, t, \ell)) \neq \emptyset$ , which contradicts the fact that  $\mathcal{L} \in \mathcal{A}(n, t, \ell)$ . Thus,  $N_+(X) \geq t + 1$  or  $N_-(X) \geq t + 1$ . ■

**Lemma 12:** Let  $\mathcal{L} \in \mathcal{A}(n, n - 2, \ell)$  be a lattice code. The word  $\mathbf{1} \in \mathbb{Z}^n$ , the all-one vector, can be covered only by a codeword of the form  $\mathbf{1} - \lambda \cdot \mathbf{e}_i$ , for some  $i$ ,  $1 \leq i \leq n$ ; where  $\lambda$  is an integer,  $0 \leq \lambda \leq \ell$ .

*Proof:* Assume that  $X \in \mathcal{L}$  is the codeword that covers  $\mathbf{1}$ . Then there exists  $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathcal{S}(n, n - 2, \ell)$  such that  $X + \mathcal{E} = \mathbf{1}$ , i.e.  $x_i = 1 - \varepsilon_i$  and therefore,  $1 - \ell \leq x_i \leq 1$  for each  $i$ ,  $1 \leq i \leq n$ . Since  $w_H(\mathcal{E}) \leq n - 2$  it follows that there are at least two entries which are equal to *one* in  $X$ . By Lemma 11, it follows that  $N_+(X) \geq n - 1$ . Hence, there are at least  $n - 1$  entries of  $X$  which are equal to *one*. Therefore,  $X = \mathbf{1} - \lambda \mathbf{e}_i$  for some  $i$ ,  $1 \leq i \leq n$ ; where  $\lambda$  is an integer,  $0 \leq \lambda \leq \ell$ . ■

**Lemma 13:** Let  $\mathcal{L} \in \mathcal{A}(n, n - 2, \ell)$  be a lattice code. For every  $j$ ,  $1 \leq j \leq n$ , the word  $W_j = \mathbf{1} - \mathbf{e}_j$  can be covered only by a codeword of the form  $\mathbf{1} - \lambda \mathbf{e}_j$ , where  $\lambda$  is an integer,  $1 \leq \lambda \leq \ell + 1$ .

*Proof:* Assume that  $X \in \mathcal{L}$  is a codeword that covers  $W_j$ . Then there exists  $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathcal{S}(n, n - 2, \ell)$  such that  $X + \mathcal{E} = W_j$ . Clearly,  $x_j = -\varepsilon_j \leq 0$ , and for each  $i \neq j$ ,  $x_i = 1 - \varepsilon_i$  and therefore  $-\ell \leq x_i \leq 1$  for each  $i$ ,  $1 \leq i \leq n$ . Since  $w_H(\mathcal{E}) \leq n - 2$  it follows that there are at most  $n - 2$  negative coordinates in  $X$ . Therefore, by Lemma 11,

it follows that  $N_+(X) \geq n - 1$ . Hence, there are at least  $n - 1$  coordinates of  $X$  which are equal to *one*. Thus,  $X = \mathbf{1} - \lambda \mathbf{e}_j$ , where  $1 \leq \lambda \leq \ell + 1$ . ■

**Lemma 14:** If there exists a perfect lattice code in  $\mathcal{A}(n, n - 2, \ell)$  then  $|\mathcal{S}(n, n - 2, \ell)|$  divides  $(\ell + 1)^{n-2}(\ell + 1 + \lambda(n - 2 - \ell))$  for some integer  $\lambda$ ,  $0 \leq \lambda \leq \ell$ .

*Proof:* Let  $\mathcal{L} \in \mathcal{A}(n, n - 2, \ell)$  be a perfect lattice code. By Lemma 12 and w.l.o.g we can assume that  $\mathbf{1}$  is covered by the codeword  $X = \mathbf{1} - \lambda \mathbf{e}_n$ , where  $0 \leq \lambda \leq \ell$ . Combining this with Lemma 13 we deduce that for all  $i$ ,  $1 \leq i \leq n - 1$ , the word  $W_i = \mathbf{1} - \mathbf{e}_i$  is covered by the codeword  $Y_i = \mathbf{1} - (\ell + 1) \cdot \mathbf{e}_i$  ( $Y_i$  cannot be equal  $\mathbf{1} - \alpha \mathbf{e}_i$ ,  $1 \leq \alpha \leq \ell$  since it would cover  $\mathbf{1}$  which is already covered by  $X$ ). We have  $n$  distinct codewords in  $\mathcal{L}$ , and since  $\mathcal{L}$  is a lattice, the lattice  $\mathcal{L}'$  generated by the set  $\{X, Y_1, Y_2, \dots, Y_{n-1}\}$  is a sublattice of  $\mathcal{L}$ , and therefore  $V(\mathcal{L}) = |\mathcal{S}(n, n - 2, \ell)|$  divides  $V(\mathcal{L}')$ . Let  $\mathbf{G}$  be the matrix whose rows are  $X, Y_1, Y_2, \dots, Y_{n-1}$ .

$$\det \mathbf{G} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 - \lambda \\ -\ell & 1 & 1 & \dots & 1 & 1 \\ 1 & -\ell & 1 & \dots & 1 & 1 \\ 1 & 1 & -\ell & \ddots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -\ell & 1 \end{vmatrix}$$

Subtracting the first row from every other row, we obtain the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 - \lambda \\ -(\ell + 1) & 0 & 0 & \dots & 0 & \lambda \\ 0 & -(\ell + 1) & 0 & \dots & 0 & \lambda \\ 0 & 0 & -(\ell + 1) & \ddots & 0 & \lambda \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(\ell + 1) & \lambda \end{vmatrix}.$$

Subtracting the first column from all the other columns, except from the last one, we obtain the determinant

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 - \lambda \\ -(\ell + 1) & \ell + 1 & \ell + 1 & \dots & \ell + 1 & \lambda \\ 0 & -(\ell + 1) & 0 & \dots & 0 & \lambda \\ 0 & 0 & -(\ell + 1) & \ddots & 0 & \lambda \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(\ell + 1) & \lambda \end{vmatrix}.$$

Finally, replacing the second row by the sum of all the rows, except for the first one, we obtain the determinant

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 - \lambda \\ -(\ell + 1) & 0 & 0 & \dots & 0 & \lambda(n - 1) \\ 0 & -(\ell + 1) & 0 & \dots & 0 & \lambda \\ 0 & 0 & -(\ell + 1) & \ddots & 0 & \lambda \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(\ell + 1) & \lambda \end{vmatrix}.$$

Now, it is easy to verify that  $V(\mathcal{L}') = |\det(\mathbf{G})| = |\lambda(n - 1)(\ell + 1)^{n-2} + (1 - \lambda)(\ell + 1)^{n-1}| = |(\ell + 1)^{n-2}(\ell + 1 + \lambda(n - 2 - \ell))|$ . ■

**Theorem 15:** There are no perfect lattice codes in  $\mathcal{A}(n, n-2, 1)$  for all  $n \geq 4$ .

*Proof:* By Lemma 14, it is sufficient to show that  $|\mathcal{S}(n, n-2, 1)| = 2^n - n - 1$  does not divide  $2^{n-2}(2 + \lambda(n-3))$ , for  $\lambda = 0, 1$ .

If  $\lambda = 0$  then we have to show that  $2^n - n - 1$  does not divide  $2^{n-1}$ . It can be readily verified that  $2^n - n - 1 > 2^{n-1}$  for all  $n > 3$ , which proves the claim.

If  $\lambda = 1$  then we have to show that  $2^n - n - 1$  does not divide  $2^{n-2}(n-1)$ . If  $2^r = \gcd(2^n - n - 1, 2^{n-2})$  then  $0 \leq r \leq \log_2(n+1)$ . Hence, we have to show that  $2^{n-r} - \frac{n+1}{2^r}$  does not divide  $n-1$ . We will show that for all  $n \geq 7$ ,  $2^{n-r} - \frac{n+1}{2^r} > n-1$ . It is easy to verify that

$$2^{n-r} - \frac{n+1}{2^r} \geq 2^{n-\log_2(n+1)} - (n+1) = \frac{2^n}{n+1} - n - 1.$$

Therefore, it is sufficient to show that

$$\frac{2^n}{n+1} - n - 1 > n - 1,$$

or equivalently

$$2^n > 2n(n+1).$$

This is simply proved by induction on  $n$  for all  $n \geq 7$ .

To complete the proof we should only verify that for  $n = 4, 5$ , and  $6$ , we have that  $2^n - n - 1$  does not divide  $2^{n-2}(n-1)$ . ■

**Theorem 16:** There are no perfect lattice codes in  $\mathcal{A}(n, n-2, \ell)$  if  $n \geq 4$  and  $\ell \geq 2$ .

*Proof:* Let  $n \geq 4$  and  $\ell \geq 2$  and assume to the contrary, that there exists a perfect lattice code  $\mathcal{L} \in \mathcal{A}(n, n-2, \ell)$ . Without loss of generality, we can assume by Lemma 12 that the word  $\mathbf{1} \in \mathbb{Z}^n$  is covered by a codeword  $X = \mathbf{1} - \lambda \mathbf{e}_n$ , where  $\lambda$  is an integer,  $0 \leq \lambda \leq \ell$ . From the proof of Lemma 14 we have that for all  $i$ ,  $1 \leq i \leq n-1$ , the word  $W_i = \mathbf{1} - \mathbf{e}_i$  is covered by the codeword  $X_i = \mathbf{1} - (\ell+1) \cdot \mathbf{e}_i$ . Therefore,  $Y = (y_1, y_2, \dots, y_n) = X_1 + X_2 = 2 \cdot \mathbf{1} - (\ell+1) \cdot \mathbf{e}_1 - (\ell+1) \cdot \mathbf{e}_2$  is a codeword. Clearly,  $y_1 = y_2 = 2 - (\ell+1) = 1 - \ell$  and since  $\ell \geq 2$  it follows that for all  $i$ ,  $1 \leq i \leq n$ ,  $|y_i| \leq \ell$ . Moreover,  $N_-(X) = 2 \leq n-2$  and  $N_+(X) = n-2$ , which contradicts Lemma 11. Thus, if  $n \geq 4$  and  $\ell \geq 2$ , then there are no perfect lattice codes in  $\mathcal{A}(n, n-2, \ell)$ . ■

Combining Theorems 15 and 16 we obtain the main result of this section.

**Corollary 6:** There are no perfect lattice codes in  $\mathcal{A}(n, n-2, \ell)$  if  $n \geq 4$  for any limited magnitude  $\ell \geq 1$ .

The existence of perfect lattice codes in  $\mathcal{A}(n, n-1, \ell)$  and their nonexistence in  $\mathcal{A}(n, n-2, \ell)$  might give an evidence that such perfect codes won't exist in  $\mathcal{A}(n, n-\epsilon, \ell)$  for  $\ell \geq 1$  and some  $\epsilon > 1$ . It would be interesting to prove such a claim for  $n \geq 4$  and  $2 \leq \epsilon \leq \lfloor \frac{n}{2} \rfloor$ .

## VII. APPLICATION TO WRITE-ONCE MEMORIES

A second possible application for a tiling of  $\mathbb{Z}^n$  with an  $n$ -dimensional chair is in constructions of multiple writing in  $n$  cells write-once memories. Each cell has  $q$  charge levels  $\{0, 1, \dots, q-1\}$ . A letter from an alphabet of size  $\sigma$ ,  $\Sigma = \{0, 1, \dots, \sigma-1\}$ , is written into the  $n$  cells as many times as possible. In each round the charge level in each cell is greater than or equal to the charge level in the previous round. It is

desired that the number of rounds for which we can guarantee to write a new symbol from  $\Sigma$  will be maximized.

An optimal solution for the problem can be described as follows. Let  $A$  be an  $q \times q \times \dots \times q$   $n$ -dimensional array. Let  $\psi : A \rightarrow \Sigma$  be a coloring of the array  $A$  with the  $\sigma$  alphabet letters. The rounds of writing and raising the charge levels of the  $n$  cells can be described in terms of the coloring  $\psi$  of the array  $A$ . If in the first round the symbol  $s_1$  is written and the charge level in cell  $i$  is raised to  $c_i^1$ ,  $1 \leq i \leq n$ , then the color in position  $(c_1^1, c_2^1, \dots, c_n^1)$  is  $s_1$ . Therefore, we have to find a coloring function  $\psi$  such that the number of rounds in which a new symbol can be written will be maximal.

Cassuto and Yaakobi [5] have found that using a coloring  $\psi$  based on a lattice tiling  $\Lambda$  with a two-dimensional chair provides the best known writing strategy when there are two cells. A coloring  $\tilde{\psi}$  of  $\mathbb{Z}^n$  based on a lattice tiling  $\Lambda$  with a shape  $\mathcal{S}$  has  $|\mathcal{S}|$  colors. The lattice have  $|\mathcal{S}|$  cosets, and hence  $|\mathcal{S}|$  coset representatives,  $X_0, X_1, \dots, X_{|\mathcal{S}|-1}$ . The points in  $\mathbb{Z}^n$  of the coset  $X_i + \Lambda$  are colored with the  $i$ -th letter of  $\Sigma$ . Now, the coloring of entry  $(x_1, x_2, \dots, x_n)$  of  $A$  given by  $\psi$  is equal to the color of the point  $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$  given by the coloring  $\tilde{\psi}$ . The method given in [5] suggests that a generalization using coloring based on tiling of  $\mathbb{Z}^n$  with an  $n$ -dimensional chair will be a good strategy for WOM codes with  $n$  cells [33]. The analysis with two cells, i.e. two-dimensional tiling was discussed with more details in [5]. The analysis for the  $n$ -dimensional case will be discussed in research work which follows by the same authors and another group as well [33].

## VIII. CONCLUSION

We have presented a few constructions for tilings with  $n$ -dimensional chairs. The tilings are based either on lattices or on generalized splitting. Both methods are equivalent if our space is  $\mathbb{Z}^n$ . The generalized splitting is a simple generalization for known concepts such as splitting and  $B_h[\ell]$  sequences. We have shown that our tilings can be applied in the design of codes which correct asymmetric errors with limited-magnitude. We further mentioned a possible application in the design of WOM codes for multiple writing. Finally, we proved that some perfect codes for correction of asymmetric errors with limited-magnitude cannot exist.

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